Grover Search and Its Cryptographic Applications

Henry Corrigan-Gibbs
Qualifying Exam Talk

21 November 2016

Quantum Computing and Crypto

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Overview

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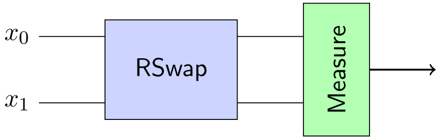

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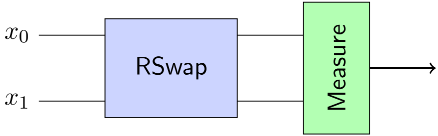

Input	$\mapsto$	*Output*
00		00
01		01 or 10
10		10 or 01
11		11

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Every possible state is a linear combination of basis states:

$$|00\rangle = \begin{pmatrix} \mathbf{1} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |01\rangle = \begin{pmatrix} 0 \\ \mathbf{1} \\ 0 \\ 0 \end{pmatrix}, \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ \mathbf{1} \\ 0 \end{pmatrix}, \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{1} \end{pmatrix}$$

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Dirac's very useful “ket” notation

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$\Rightarrow$ Computation is just a matrix-vector product.

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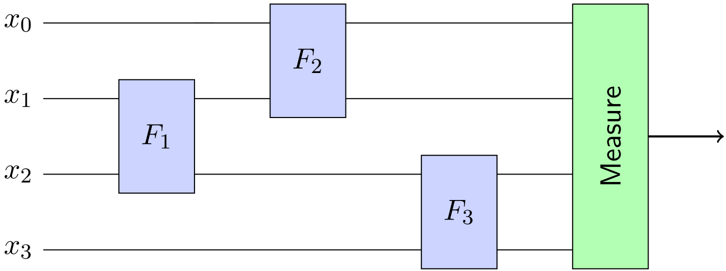
Example: Quantum Circuit



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2. Amplitudes can be **negative**, unlike probabilities.
   – This is the source of QC's apparent power.

Useful Tool: Hadamard Gate

Definition

The *Hadamard gate* H is the quantum analogue of a classical bit-flip:

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The operator $H^{\otimes n}$ applies H to each of n qubits.

Useful Tool: Quantum Queries

Fact (Lecerf 1963, Bennett 1973)

If $f : \{0,1\}^n \to \{0,1\}$ *is computable with a* $T(n)$*-size classical circuit, then there is a size-*$O(T(n))$ *quantum circuit that maps:*

$$|x\rangle|y\rangle \quad \mapsto \quad |x\rangle|y \oplus f(x)\rangle,$$

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Can make quantum queries to a classical function!

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This essentially changes the sign of "good" xs in a superposition.

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Fact

A classical algorithm for unstructured search that succeeds with constant probability must make $\Omega(N)$ *queries.*

Theorem (Grover 1996)

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There is a **quantum** algorithm for unstructured search that makes $O(\sqrt{N})$ **quantum** queries and succeeds with probability at least $2/3$.

Grover's Algorithm

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The Algorithm.

1. **Initialize** an n-bit register to the state $H^{\otimes n}|0^n\rangle$.
2. **Apply** the following operator $O(\sqrt{N})$ times:

$$G = -H^{\otimes n} Q_0 H^{\otimes n} Q_f.$$

3. **Measure** the state of the register and output it.

Analysis of Grover's Algorithm

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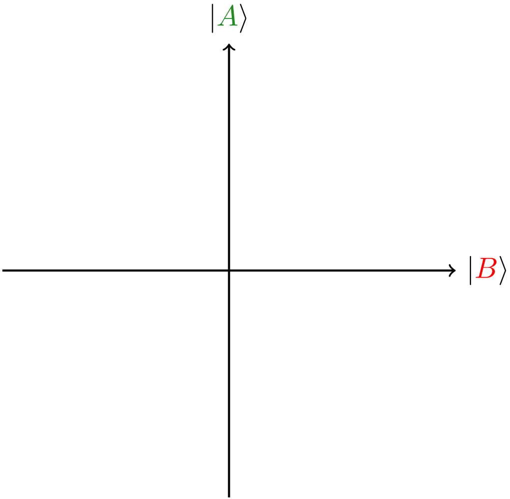
After initialization, the register is in the uniform superposition over strings:

$$H^{\otimes n}|0^n\rangle = |h\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle = \underbrace{\sqrt{\frac{a}{N}}|A\rangle}_{\text{Awesome}} + \underbrace{\sqrt{\frac{b}{N}}|B\rangle}_{\text{Bad}}$$

Analysis of Grover's Algorithm

$$G = -H^{\otimes n} Q_0 H^{\otimes n} Q_f$$



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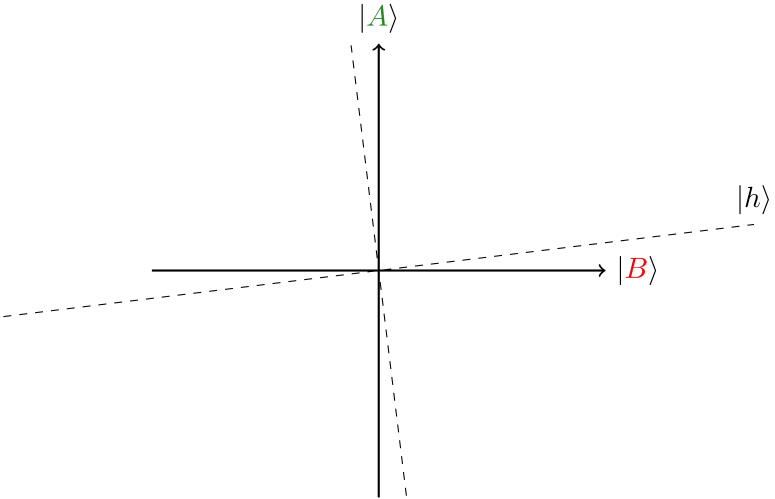

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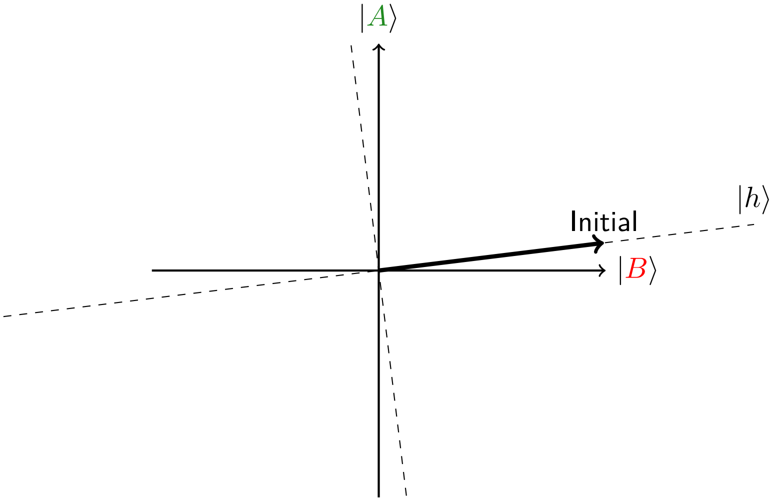

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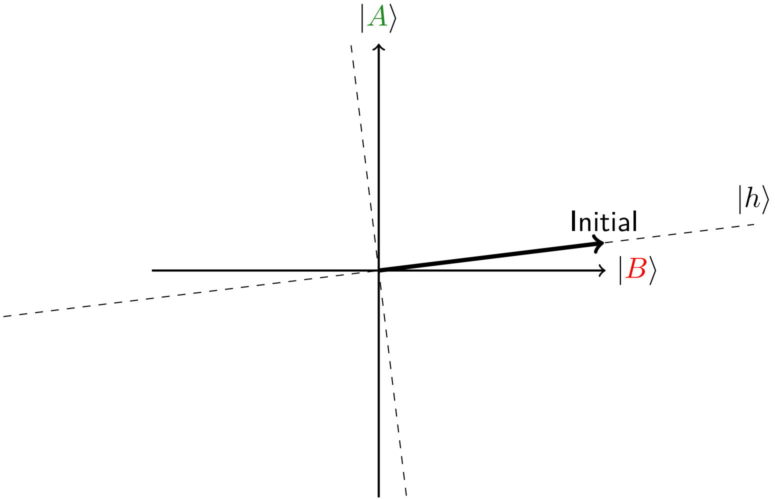

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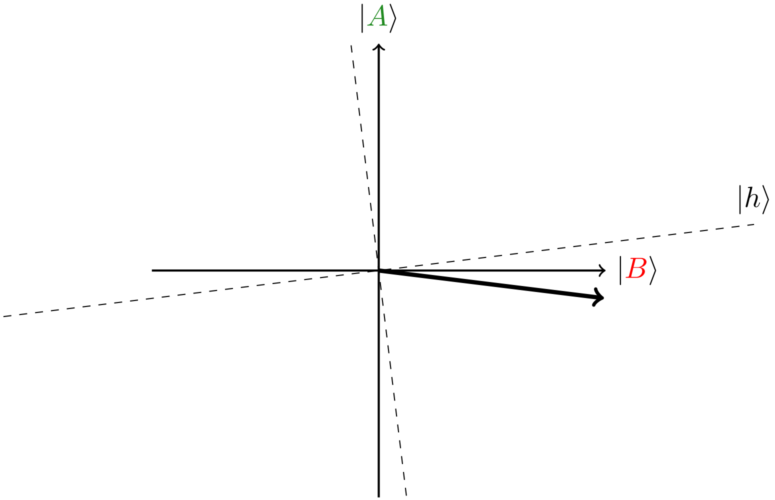

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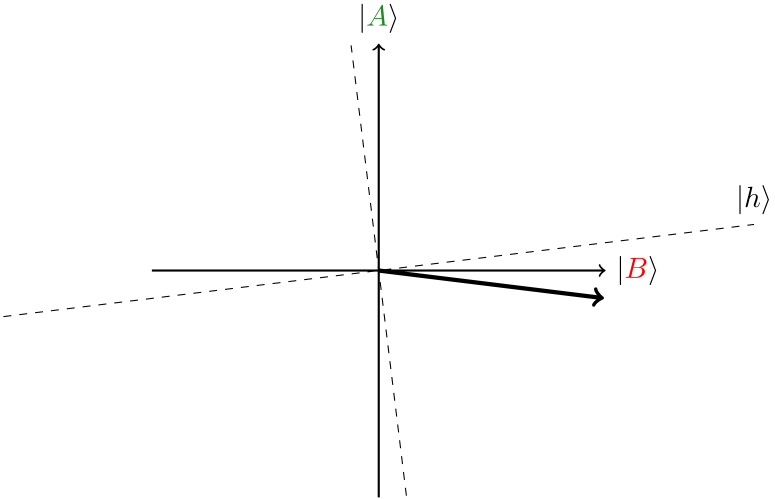

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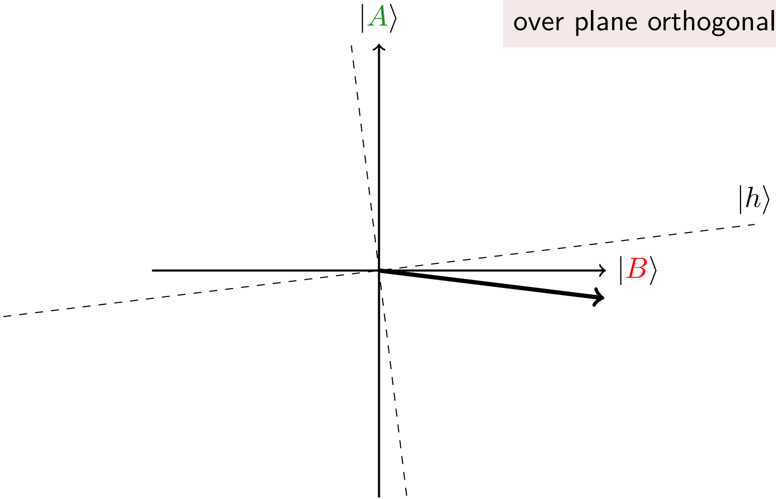

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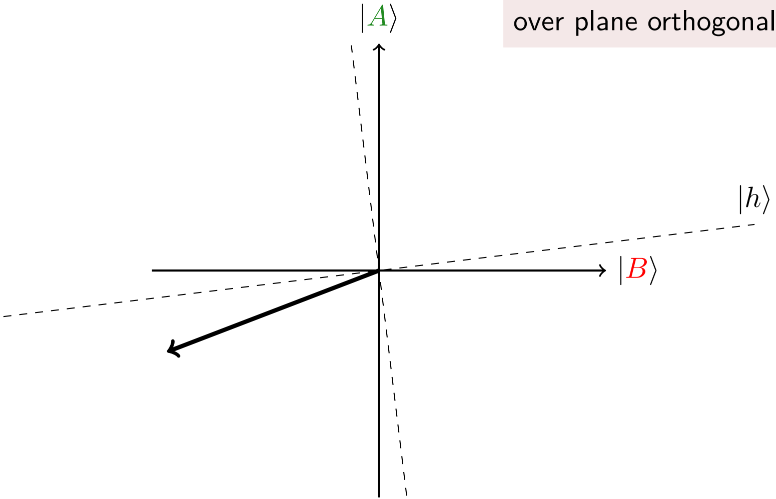

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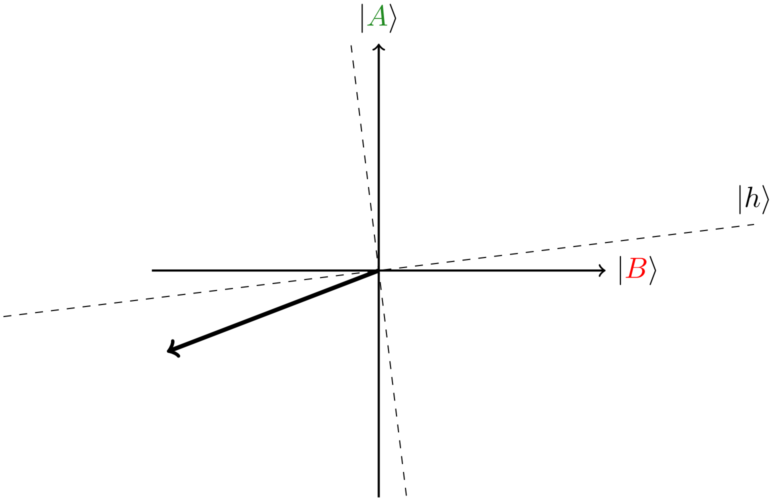

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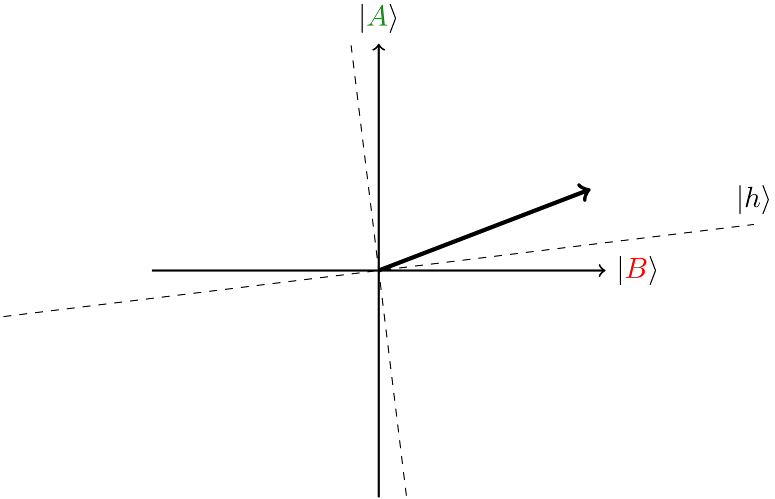

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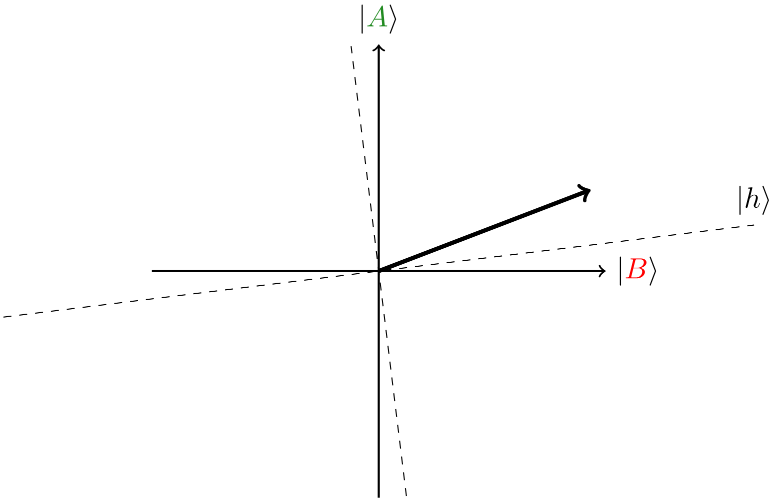

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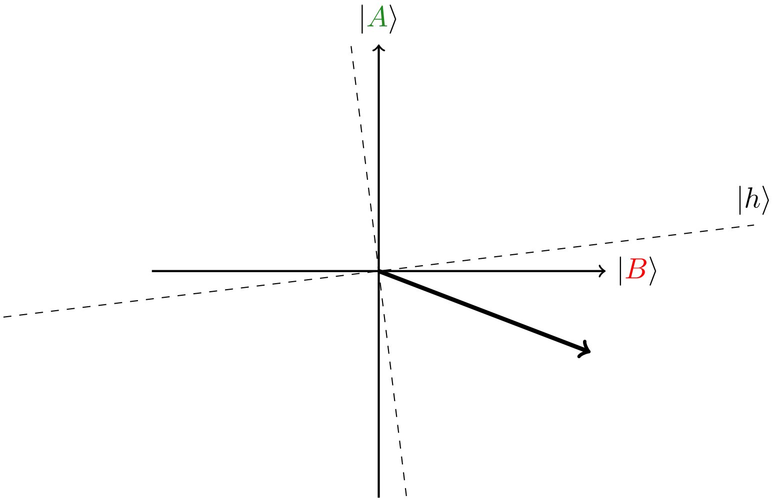

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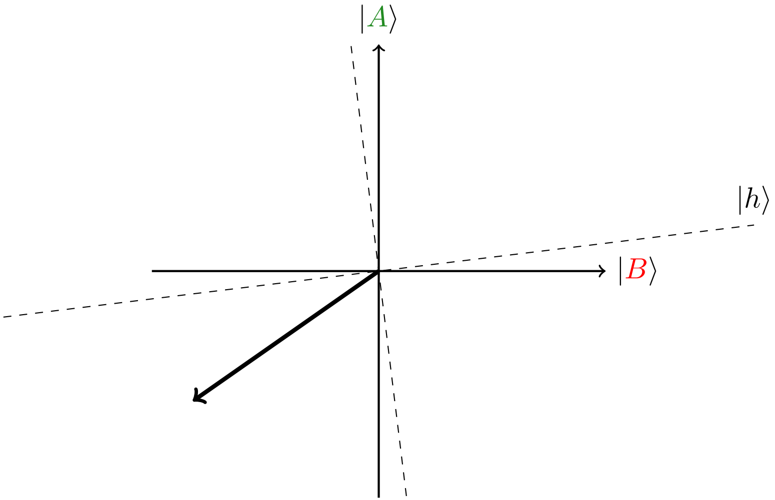

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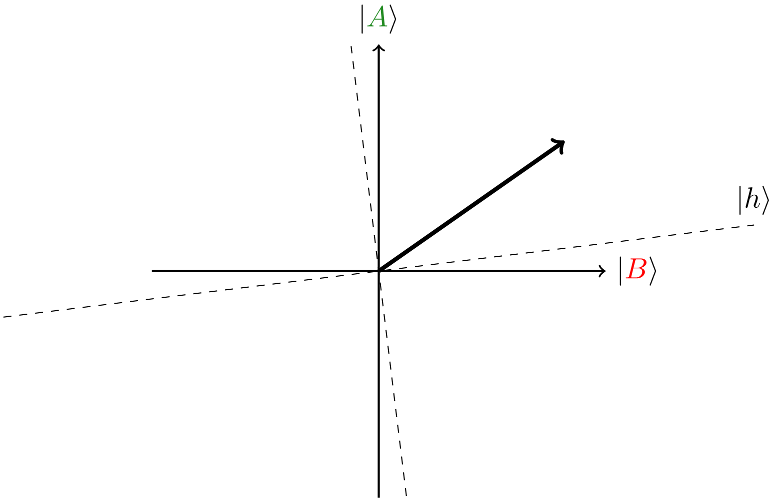

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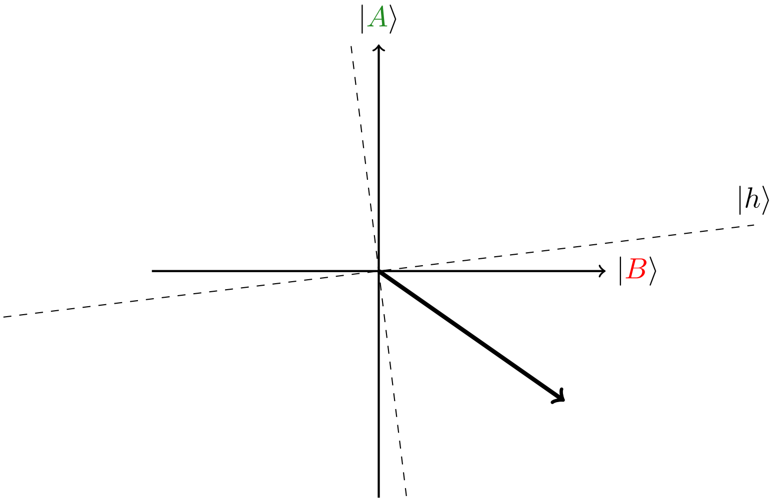

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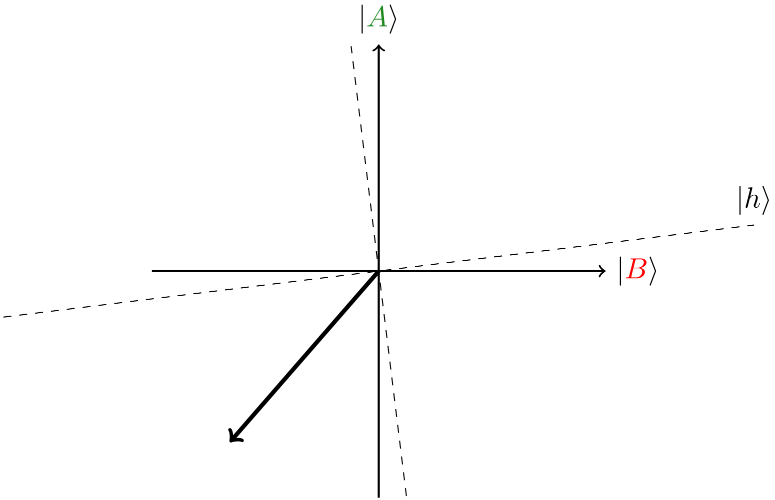

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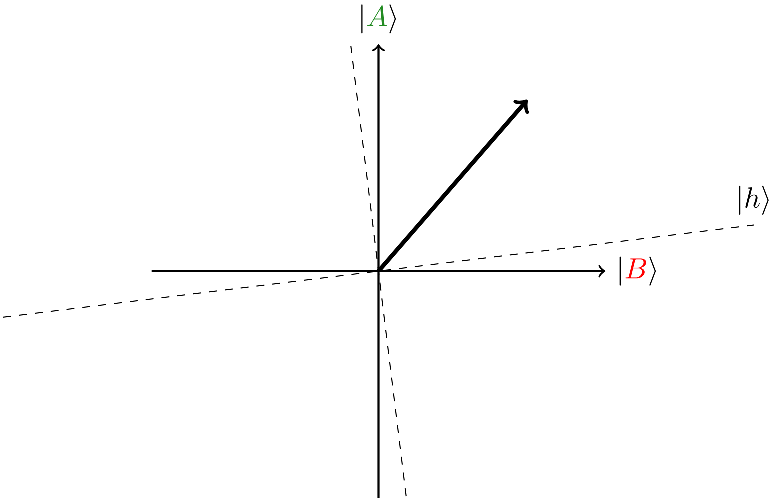

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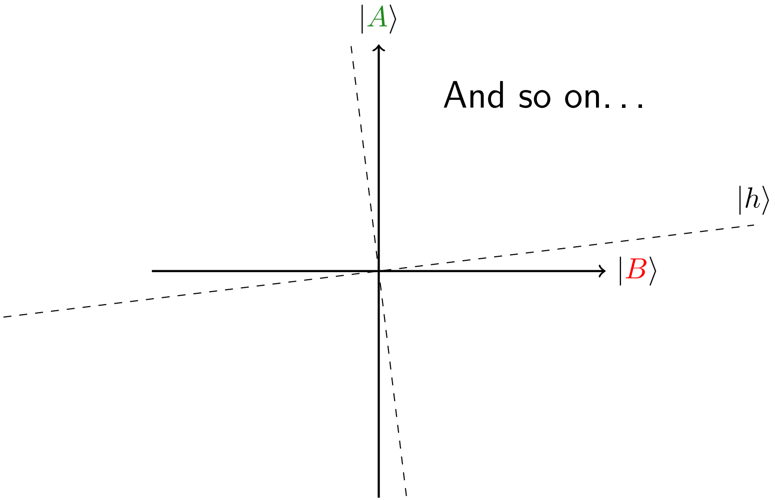

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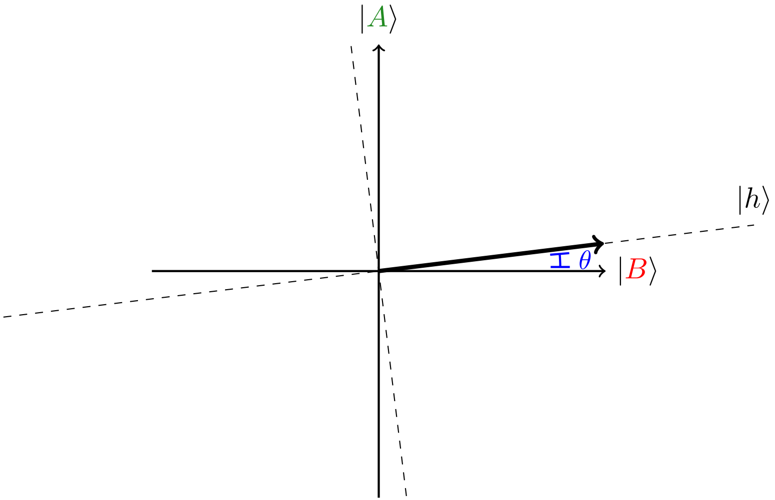

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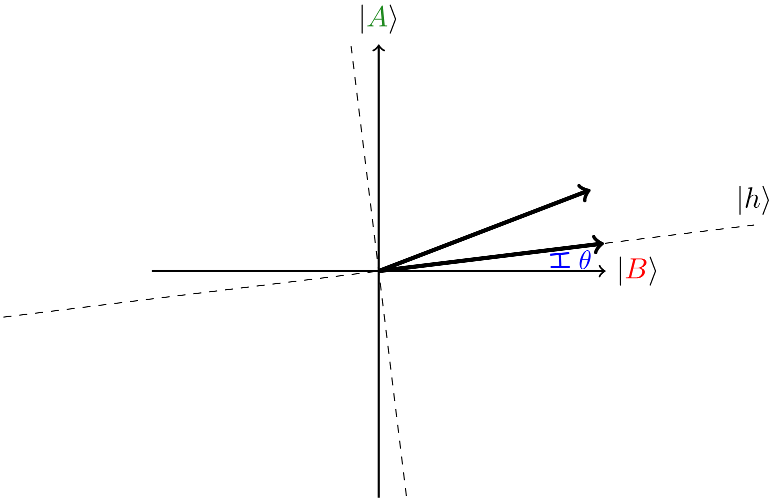

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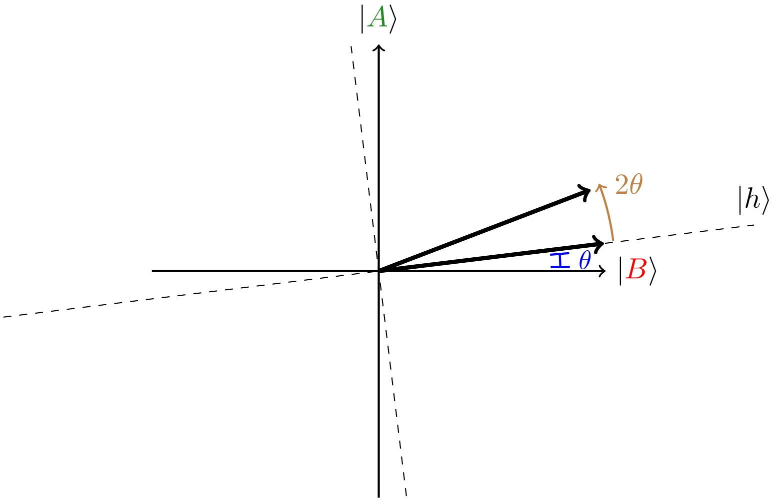

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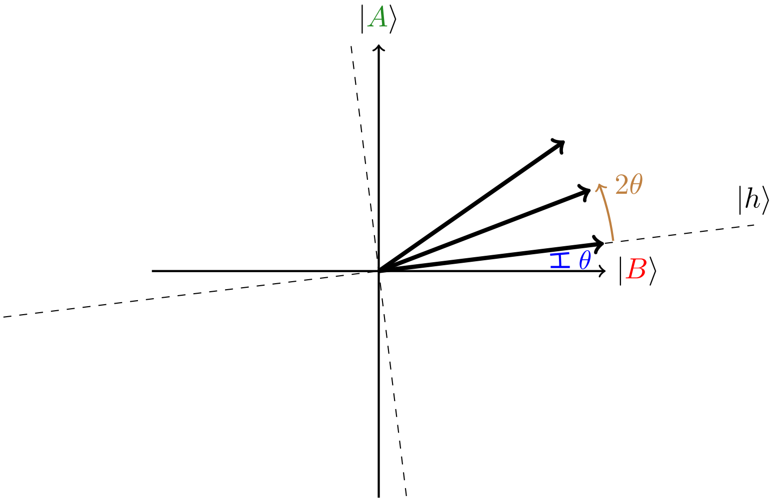

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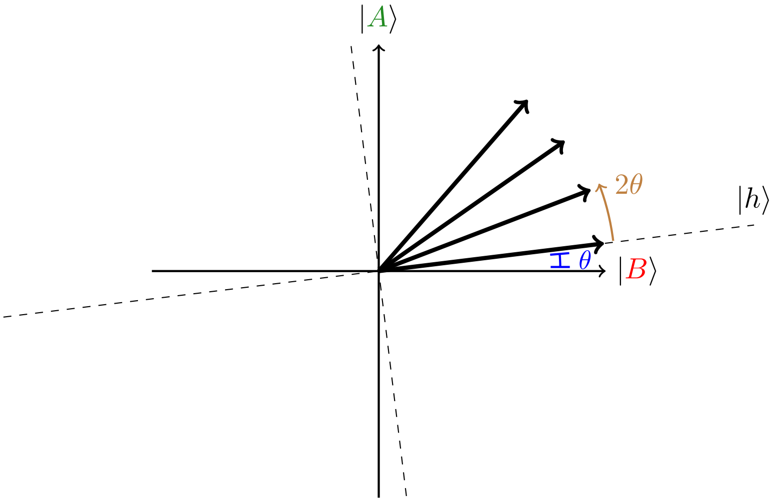

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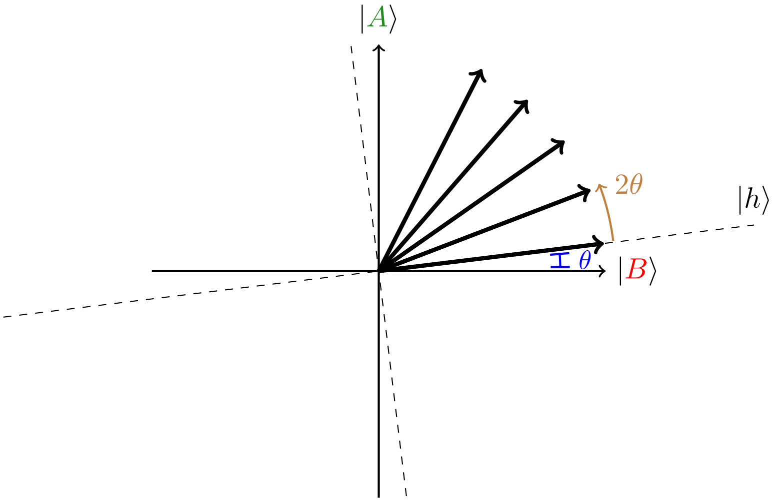

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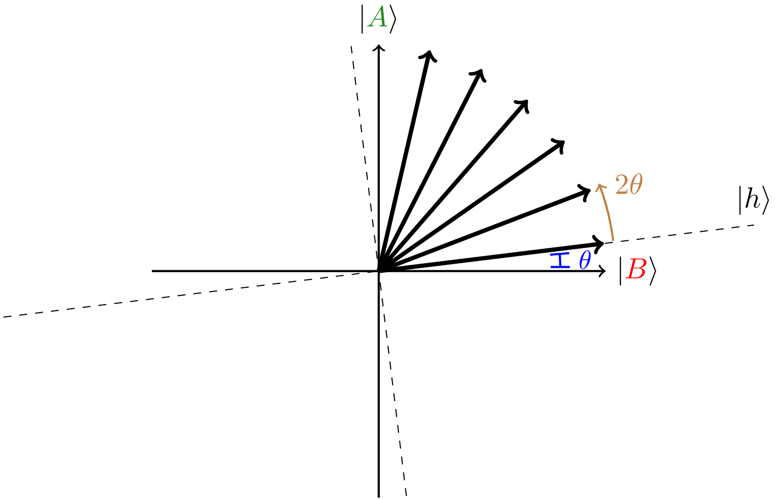

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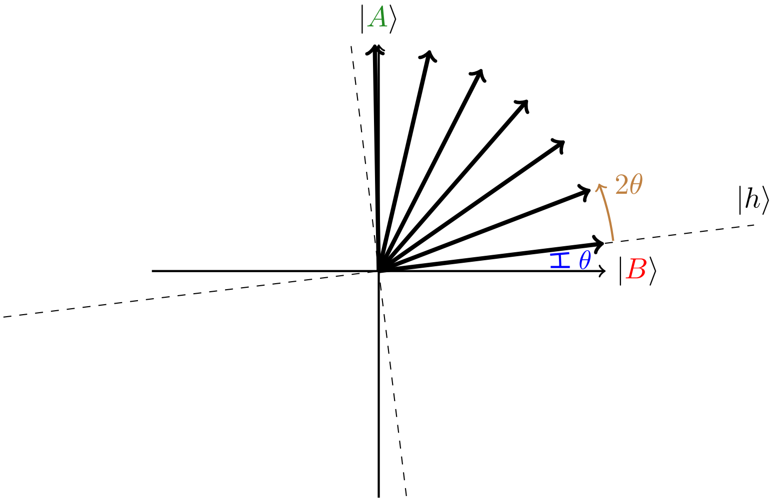
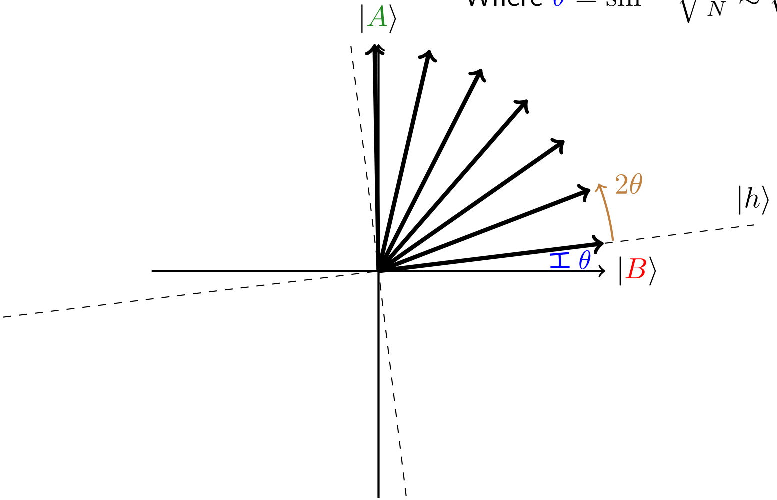


Analysis of Grover's Algorithm

Where $\theta = \sin^{-1}\sqrt{\frac{a}{N}} \approx \sqrt{\frac{a}{N}}$



Analysis of Grover's Algorithm

After t Grover iterations, the angle between the register state and $|B\rangle$ is $\approx 2\theta t$. We want the bad state $|B\rangle$ and the register state to be orthogonal:

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Num. Solutions	Iterations
1	$\frac{\pi}{4} \cdot \sqrt{N}$
a	$\frac{\pi}{4} \cdot \sqrt{\frac{N}{a}}$
Unknown	$t \leftarrow_R \{1, \ldots, \sqrt{N}\}$

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One query per iteration $\Rightarrow O(\sqrt{N})$ queries.

Lower Bound

Definition (Decision Grover Problem)

Given oracle access to $f : [N] \to \{0, 1\}$, decide whether there exists an x such that $f(x) = 1$ with probability better than 2/3.

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Theorem (Bennet, Bernstein, Brassard, Vazirani 1997)

For every quantum algorithm that makes $o(\sqrt{N})$ queries to f, there exists an f for which the algorithm fails to solve the Decision Grover Problem.

Thm. For every quantum algorithm that makes $o(\sqrt{N})$ queries to f, there exists an f for which the algorithm fails to solve the DGP.

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Proof Idea. Fix a T-query quantum algorithm:

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$$\Rightarrow \quad \frac{\epsilon}{2}\sqrt{N} \leq T$$

Overview

Motivation

Background

Grover's Algorithm

Applications
- Breaking Block Ciphers
- Collision Finding
- Password Cracking

Conclusion

Breaking Block Ciphers

For this talk, a block cipher is an efficient deterministic function:

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cannot recover k faster than a brute-force search of the key-space.

Viewing $E(\cdot,\cdot)$ as an oracle, an adversary making q queries should succeed with probability at most $\approx q/|\mathcal{K}|$.

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3. Run Grover's algorithm on f_c.
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Attacking AES-128

Special-purpose classical attack: $2^{126.1}$ (Bogdanov et al. 2011)

Breaking Block Ciphers

Grover search recovers the key in time $O(\sqrt{|\mathcal{K}|})$.

Attack Using Grover

1. Attacker receives challenge $c = (c_0, c_1, c_2)$.
2. Define a function $f_c : \mathcal{K} \to \{0,1\}$ as:

$$f_c(k) \stackrel{\text{def}}{=} \{(E(k, \text{“0”}), E(k, \text{“1”}), E(k, \text{“2”})) = (c_0, c_1, c_2)\}.$$

3. Run Grover's algorithm on f_c.
4. In $O(\sqrt{|\mathcal{K}|})$ iterations, Grover returns k w.h.p.

Attacking AES-128

Special-purpose classical attack: $2^{126.1}$ (Bogdanov et al. 2011)
Generic quantum attack: 2^{64}. **!!!**

Hash Collisions

Let H be a random function.

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[Compute $H(0), H(1), H(2), \ldots$ until you find a collision.]

Theorem (Brassard, Høyer, Tapp 1997)

There is a quantum collision-finding algorithm that makes $O(N^{1/3})$ quantum queries and succeeds with constant probability.

Quantum Collision Finding

Algorithm Idea

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- Build a big table of random values and their hashes.

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r_3	$H(r_3)$
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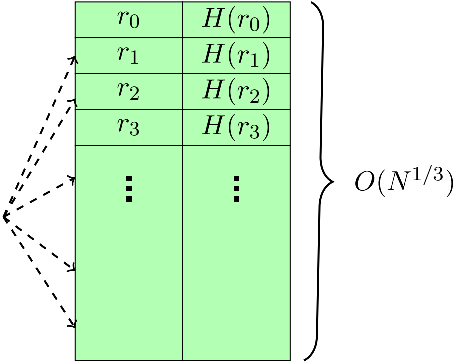

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Collision Finding in Practice

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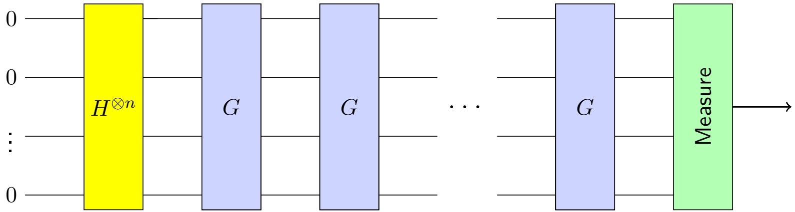

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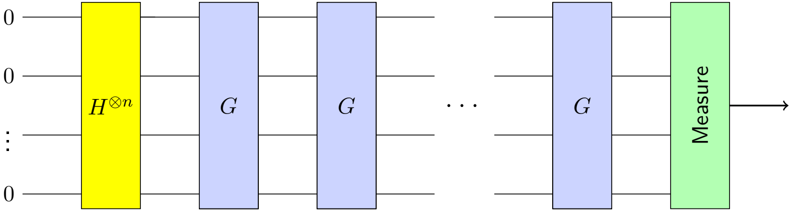

Each Grover iteration encodes a table of size $\Theta(N^{1/3})$, so the G circuit has $\Theta(N^{1/3})$ gates. (!)

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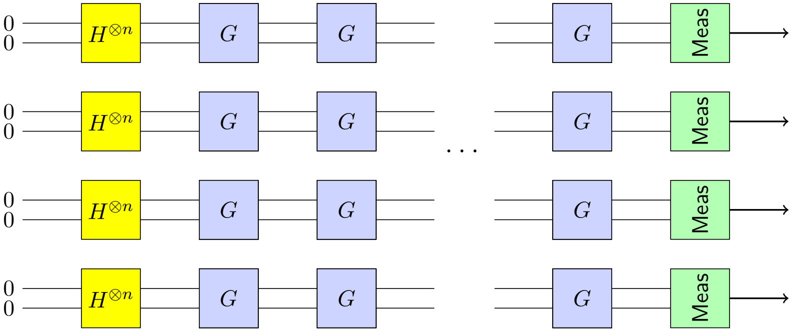

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Parallel Grover (Grover and Rudolph 2003)

1. Pick an $x_0 \xleftarrow{R} [N]$.
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If you have a size-$\Theta(N^{1/3})$ **classical** computer, finding collisions with the parallel rho method only takes time $O(N^{1/6})$!
(Van Oorschot and Wiener 1999) (Bernstein 2009)

Password Cracking

Modern OSes store passwords as $H(\texttt{salt}, \texttt{password})$, where:

- H is a "moderately hard" function, and
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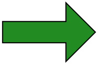

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If someone steals your password file, they have to do some work ("password cracking") to recover the stored passwords.

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Problem: Given oracle access to $H : [N] \to [N]$, a dictionary of candidate passwords

$$\mathcal{D} = \{\texttt{password, 12345, qwerty, ...}\} \subseteq [N],$$

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Inverting a function with *hints*.

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Quantum computers essentially break all password hashing functions.

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This often beats the classical $|\mathcal{D}| \cdot \mathcal{C}_H$ attack!

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If we can represent the dictionary $\mathcal{D}$ with a **small circuit**, then the quantum attack is devastating:

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Using **amplitude amplification** (Brassard, Høyer, Mosca, Tapp 2002), we can generalize the attack from

password dictionaries to *password distributions*.

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Type	Len	Classical	Quantum
Lower-case alpha	6 char	2^{28}	2^{14}
	8 char	2^{37}	2^{19}
	10 char	2^{47}	2^{24}
Alphanumeric	6 char	2^{36}	2^{18}
	8 char	2^{47}	2^{23}
	10 char	2^{60}	2^{30}
Printable ASCII	6 char	2^{39}	2^{20}
	8 char	2^{52}	2^{26}
	10 char	2^{66}	2^{33}

Overview

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Thank you!

References

Background

- Sanjeev Arora and Boaz Barak. *Computational Complexity: A Modern Approach.*
- Michael Nielsen and Isaac Chuang. *Quantum Computation and Quantum Information.*
- John Watrous. Lecture notes: *Introduction to Quantum Computing* `https://cs.uwaterloo.ca/~watrous/LectureNotes.html`

Grover's Algorithm

- Lov Grover. "A Fast Quantum Mechanical Algorithm for Database Search" (1996).
  `https://arxiv.org/abs/quant-ph/9605043`
- Michel Boyer, Gilles Brassard, Peter Høyer, and Alain Tapp. "Tight Bounds on Quantum Searching" (1996).
  `https://arxiv.org/abs/quant-ph/9605034`
- Richard Jozsa. "Searching in Grover's Algorithm" (1999).
  `https://arxiv.org/pdf/quant-ph/9901021`
  Gives the nice geometric interpretation of Grover search.
- Gilles Brassard, Peter Høyer, Michele Mosca, and Alain Tapp. "Quantum Amplitude Amplification and Estimation" (2000).
  `https://arxiv.org/abs/quant-ph/0005055`

References

Lower Bound

- Charles H. Bennett, Ethan Bernstein, Gilles Brassard, and Umesh Vazirani "Strengths and Weaknesses of Quantum Computing" (1997). `https://arxiv.org/abs/quant-ph/9701001`
- Ronald de Wolf. Lecture notes: "Quantum Lower Bounds" (2005). `http://www.iro.umontreal.ca/~tappa/Summer%20School/montreal05.pdf`
- Scott Aaronson. Lecture notes: "6.845: Quantum Complexity Theory" (2009). `https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-845-quantum-complexity-theory-fall-2010/lecture-notes/`

Collision Finding

- Gilles Brassard, Peter Høyer, and Alain Tapp. "Quantum Algorithm for the Collision Problem" (1997). `https://arxiv.org/abs/quant-ph/9705002`
- Paul van Oorschot and Michael J. Wiener. "Parallel Collision Search with Cryptanalytic Applications" (1999). `http://people.scs.carleton.ca/~paulv/papers/JoC97.pdf`
- Lov Grover and Terry Rudolph. "How Significant are the Known Collision and Element Distinctness Quantum Algorithms?" (2003). `http://arxiv.org/pdf/quant-ph/0306017`
- Daniel J. Bernstein. "Cost Analysis of Hash Collisions: Will Quantum Computers Make SHARCs Obsolete?" (2009). `http://cr.yp.to/hash/collisioncost-20090823.pdf`

References

AES Cryptanalysis

- Andrey Bogdanov, Dmitry Khovratovich, and Christian Rechberger. "Biclique Cryptanalysis of the Full AES" (2011). `http://research.microsoft.com/en-us/projects/cryptanalysis/aesbc.pdf`

Analysis of Grover's Algorithm

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Then $R = H^{\otimes n} Q_0 H^{\otimes n} = I - 2|h\rangle\langle h|$, so R takes:

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So, for any vector $|v\rangle = \alpha|h\rangle + \beta|h^\perp\rangle$, R maps:

$$\alpha|h\rangle + \beta|h^\perp\rangle \qquad \mapsto \qquad -\alpha|h\rangle + \beta|h^\perp\rangle.$$

$\square$